Lecture 10

- Fuchsian Differential Equations
- Hypergeometric functions
- Confluent Hypergeometric functions

Fuchsian DE

(i) Fuchsian DE with two regular singular point.

(ii) Fuchsian DE with three regular singul point.
(Hypergeometric Function)

(iii) Fuchsian DE with two regular singul point
one of order two.
(Confluent hypergeometric functions)

$$u'' + \frac{a_0}{z-z_1} u'' + \frac{b_0}{(z-z_1)^2} u = 0$$ Euler type.

$z_1 \neq 0$.

Fuchsian Differential Equations

Definition 1: A homogeneous DE with single-valued analytic coefficient is called "Fuchsian" differential DE if it has only regular singular points in the extended complex plane, i.e., the complex plane including the point at infinity.

Definition 2 (behavior at infinity). Let $z = 1/t$

$u'' + p(z)u' + q(z)u = 0$ changes to, with $v(t) = u(1/t)$

$v'' + [\frac{2}{t} - \frac{1}{t^2} r(t)]v' + \frac{1}{t^4} s(t) v = 0$ where $r(t) = p(1/t)$

$s(t) = q(1/t)$. ~~Proo~~

~~$u_z\ t \cdot v_t\ \frac{1}{t^2}\ v,\ u_{zz} = v_{tt}\ \frac{1}{t^4}$~~

$u_z = -v_t \frac{1}{z^2}, \quad u_{zz} = \frac{2v_t}{z^3} + v_{tt}\frac{1}{z^4} = 2t^3 v_t + t^4 v_{tt}$

$2t^3 \cancel{v_t} + t^4 v_{tt} - p(1/t) v_t t^2 + q(1/t) v = 0$

$v_{tt} + [\frac{2}{t} - \frac{1}{t^2} r(t)] v_t + \frac{1}{t^4} s(t) v = 0$

We assume that $t=0$ is a regular singular point hence

$r(t) = a_1 t + a_2 t^2 + \dots$

$s(t) = b_2 t^2 + b_3 t^3 + \dots$

$p(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$

$q(z) = \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$

Proposition 1. A second order Fuchsian DE with two regular singular points can be reduced to a DE with constant coefficients.

Proof: let $z = z_1$ and $z = z_2$ be the regular singular points of the DE. Define now

$$\xi = \frac{z - z_1}{z - z_2}$$

then the DE (1) reduces to

$$u_{\xi\xi} + P(\xi)\, u_\xi + Q(\xi)\, u = 0$$

$$u_z = u_\xi \left(\frac{z - z_2 - z + z_1}{(z - z_2)^2} \right) = u_\xi \, \frac{(z_1 - z_2)}{(z - z_2)^2}$$

$$u_{zz} = u_{\xi\xi} \, \frac{(z_1 - z_2)^2}{(z - z_2)^4} - 2 u_\xi \, \frac{z_1 - z_2}{(z - z_2)^3}$$

$$u_{\xi\xi} \, \frac{(z_1 - z_2)^2}{(z - z_2)^4} - 2 u_\xi \, \frac{z_1 - z_2}{(z - z_2)^3} + p(z) \, \frac{z_1 - z_2}{(z - z_2)^2} u_\xi + q(z)\, u = 0$$

$$u_{\xi\xi} + P(\xi)\, u_\xi + Q(\xi)\, u = 0$$

$$\frac{(z - z_2)^2}{(z_1 - z_2)^2} \left[-2 \, \frac{z_1 - z_2}{z - z_2} + p\,(z_1 - z_2) \right] = P(\xi)$$

$$\frac{(z - z_2)^4}{(z_1 - z_2)^2} \, q(z) = Q(\xi)$$

$$P(\xi) = -2\frac{z-z_2}{z_1-z_2} + \frac{(z-z_2)^2 p(z)}{z_1-z_2}$$

$$z\xi - \xi z_2 = z - z_1$$

$$z(\xi-1) = \xi z_2 - z_1 \qquad z = \frac{\xi z_2 - z_1}{\xi - 1}$$

$$z = z_2 \ : \ \xi = \infty$$

$$z = z_1 \ : \ \xi = 0$$

$$z - z_2 = \frac{\xi z_2 - z_1}{\xi - 1} - z_2 = \frac{\xi z_2 - z_1 - z_2\xi + z_2}{\xi - 1} = \frac{z_2 - z_1}{\xi - 1}$$

$$P(\xi) = \frac{2}{\xi - 1} + (z_1 - z_2)\frac{1}{(\xi-1)^2} p$$

$$Q(\xi) = \frac{(z_1-z_2)^2}{(\xi-1)^4} q(z)$$

$$u_{\xi\xi} + P(\xi) u_\xi + Q(\xi) u = 0$$

$\xi = \infty$ is the regular singular point

$$P(\xi) = \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \cdots$$

$$Q(\xi) = \frac{b_2}{\xi^2} + \frac{b_3}{\xi^3} + \cdots$$

$\xi = 0$ is also regular singul. $\qquad \xi P(\xi) = $ analy

$\qquad \xi^2 Q(\xi) = $ analy

$a_2 = 0, \quad a_3 = 0 \quad \cdots \qquad u_{\xi\xi} + \frac{a_1}{\xi} u_\xi + \frac{b_2}{\xi^2} u = 0 \quad$ Euler type.

$b_3 = 0, \quad b_4 = 0 \cdots$

Regular singular points with three regular singl. Points

$$u'' + \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3} \right] u'$$

$$+ \left[\frac{(z_1-z_2)(z_1-z_3)}{z-z_1}\alpha\alpha' + \frac{(z_2-z_1)(z_2-z_3)}{z-z_2}\beta\beta' + \right.$$

$$\left. + \frac{(z_3-z_1)(z_3-z_2)}{z-z_3}\gamma\gamma' \right] \frac{u(z)}{(z-z_1)(z-z_2)(z-z_3)} = 0$$

$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$ "Riemann equation"

$z = z_1 \quad : \quad r_1 = \alpha, \; r_2 = \alpha'$

$z = z_2 \quad : \quad r_1 = \beta, \; r_2 = \beta'$

$z = z_3 \quad : \quad r_1 = \gamma, \; r_2 = \gamma'$

$$u(z) = P \begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$ "Riemann P symbol"

The 9 parameters which characterize the Riemann equation. These parameters can be reduced to 3 by making use of the following transformations on the equation

(i) $u(z) = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, v(z)$

with $r+s+t=0$

(ii) $z' = \dfrac{Az+B}{Cz+D}$

<u>Proof.</u>

$$u' = r(z-z_1)^{r-1}(z-z_2)^s(z-z_3)^t v$$
$$+ s(z-z_1)^r(z-z_2)^{s-1}(z-z_3)^t v$$
$$+ t(z-z_1)^r(z-z_2)^s(z-z_3)^{t-1} v$$
$$+ (z-z_1)^r(z-z_2)^s(z-z_3)^t v'$$

$$u'' = r(r-1)(z-z_1)^{r-2}(z-z_2)^s(z-z_3)^t v$$
$$+ s(s-1)(z-z_1)^r(z-z_2)^{s-2}(z-z_3)^t v$$
$$+ t(t-1)(z-z_1)^r(z-z_2)^s(z-z_3)^{t-2} v$$
$$+ 2rs(z-z_1)^{r-1}(z-z_2)^{s-1}(z-z_3)^t v.$$
$$+ 2rt(z-z_1)^{r-1}(z-z_2)^s(z-z_3)^{t-1} v$$
$$+ 2st(z-z_1)^r(z-z_2)^{s-1}(z-z_3)^{t-1} v$$
$$+ 2r(z-z_1)^{r-1}(z-z_2)^s(z-z_3)^t v'$$
$$+ 2s(z-z_1)(z-z_2)^{s-1}(z-z_3)^t v'$$
$$+ 2t(z-z_1)(z-z_2)^s(z-z_3)^{t-1} v'$$
$$+ (z-z_1)^r(z-z_2)^s(z-z_3)^t v''$$

$$u'' + \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3}\right] u'$$

(6)

$$v'' + \left[\frac{2\cancel{r}}{z-z_1} + \frac{2s}{z-z_2} + \frac{2t}{z-z_3} \right] v'$$

$$+ \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3} \right] v'$$

$$+ \frac{r(r-1)}{(z-z_1)^2} + \frac{s(s-1)}{(z-z_2)^2} + \frac{t(t-1)}{(z-z_3)^2} + \frac{2rs}{(z_1-z_1)(z-z_2)}$$

$$+ \frac{2r\cancel{t}}{(z-z_1)(z-z_3)} + \frac{2st}{(z-z_2)(z-z_3)} \quad |$$

$$+ \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3} \right] \left(\frac{r}{z-z_1} + \frac{s}{z-z_2} + \frac{t}{z-z_3} \right) v$$

$$+ \left[\qquad \qquad \right] \frac{v}{(z-z_1)(z-z_2)(z-z_3)} = 0$$

$$v'' + \left[\frac{1-(\alpha\mp r)-(\alpha'\mp r)}{z-z_1} + \frac{1-(\beta\mp r)-(\beta'\mp r)}{z-z_2} + \frac{1-(\gamma\mp r)-(\gamma'\mp r)}{z-z_3} \right] v$$

$$v(z) = P \begin{Bmatrix} z_1 & z_2 & z_3 \\ \alpha\mp r & \beta\mp r & \gamma\mp r \\ \alpha'\mp r & \beta'\mp r & \gamma'\mp r \end{Bmatrix}$$

$$\Rightarrow \quad P \begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = (z-z_1)^r (z-z_2)^s (z-z_3)^t P \begin{Bmatrix} z_1 & z_2 & z_3' \\ \alpha\cancel{+}r & \beta\cancel{+}s & \gamma\cancel{+}t\, z \\ \alpha'\cancel{+}r & \beta'\cancel{+}s & \gamma'\cancel{+}t \end{Bmatrix}$$

$$= (z-z_1)^r (z-z_2)^s (z-z_3)^t P \begin{Bmatrix} z_1 & z_2 & z_3 \\ \alpha-r & \beta-s & \gamma-t \\ \alpha'-r & \beta'-s & \gamma'-t \end{Bmatrix}$$

or

$$P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha-r & \beta-s & \gamma-t & z \\ \alpha'-r & \beta'-s & \gamma'-t & \end{matrix}\right\}$$

$(r \to -r \;,\; s \to -s \;,\; t \to -t)$

or

$$P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha+r & \beta+s & \gamma+t & z \\ \alpha'+r & \beta'+s & \gamma'+t & \end{matrix}\right\} = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

$$r+s+t=0$$

(ii) by using the Mobius transformation $z' = \dfrac{Az+B}{Cz+D}$ with $AD-BC \neq 0$ then

$$P\left\{\begin{matrix} z_1' & z_2' & z_3' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

where $\quad z_i' = \dfrac{Az_i+B}{Cz_i+D} \quad , \; i = 1,2,3.$

let z_1', z_2' and z_3' be respectively $0, \infty$ and 1.

hence $z_1' = 0$ $\quad 0 = Az_1 + B$, $\quad \dfrac{z_2' = \infty}{Cz_2 + D = 0}$, $\quad 1 = \dfrac{Az_3 + B}{Cz_3 + D}$

$\dfrac{C}{D} = -\dfrac{1}{z_2}$, $\quad B = -Az_1$,

$z_3' = 1$ $\quad 1 = \dfrac{Az_3 - Az_1}{Cz_3 + D} = \dfrac{A}{D}\left(\dfrac{z_3 - z_1}{1 - z_3/z_2}\right)$

$\rightarrow \quad \dfrac{A}{D} = \dfrac{z_2 - z_3}{z_2(z_3 - z_1)}$, $\rightarrow \dfrac{C}{D} = -\dfrac{1}{z_2}$

$\rightarrow \quad \dfrac{B}{D} = -\dfrac{A}{D} z_1 = -\dfrac{z_1(z_2 - z_3)}{z_2(z_3 - z_1)}$

$\Downarrow$

$\Rightarrow \quad z' = \dfrac{Az + B}{Cz + D} = \dfrac{\frac{A}{D} z + \frac{B}{D}}{\frac{C}{D} z + 1} = \dfrac{(z_2 - z_3) z - z_1 (z_2 - z_3)}{-(z_3 - z_1) z + z_2 (z_3 - z_1)}$

$$z' = \frac{(z_2 - z_3)(z - z_1)}{(z_3 - z_1)(-z + z_2)}$$

Choosing r, s, t properly we simplify the solution

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = (z-z_1)^{-r}(z-z_2)^{-s}(z-z_3)^{-t} P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha+r & \beta+s & \gamma+t & z \\ \alpha'+r & \beta'+s & \gamma'+t & \end{Bmatrix}$$

$$r = -\alpha, \quad s = \alpha+\gamma, \quad t = -\gamma$$

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha, & \beta, & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = (z-z_1)^{\alpha}(z-z_2)^{-\alpha-\gamma}(z-z_3)^{\gamma} P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ 0 & \overset{(a)}{\alpha+\beta+\gamma} & 0 & z \\ \underset{(1-c)}{\alpha'-\alpha} & \overset{(b)}{\alpha+\beta'+\gamma} & \underset{(c-a-b)}{\gamma'-\gamma} & \end{Bmatrix}$$

$a = \alpha+\beta+\gamma$
$b = \alpha+\beta'+\gamma$
$1-c = \alpha'-\alpha$
$c-a-b = \gamma'-\gamma$
$1 = 1$

$z \to z'$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0 & z' \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

where

$$z' = \frac{(z_3-z_2)(z-z_1)}{(z_3-z_1)(z-z_2)}$$

let $\quad a = \alpha+\beta+\gamma, \quad b = \alpha+\beta'+\gamma, \quad c = 1+\alpha-\alpha'$

$$\Rightarrow P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1-c & b & c-a-b & \end{Bmatrix}$$

0) Rieman Eqn $\to$ hypergeometric eqn

1) Solutions about the singular points

2) Kummer's Solution

3) Find the solutions about $z=1$ and $z=\infty$ by using the Kummer's solutions.

$$\Phi\left\{\begin{matrix} z_1 & z_2 & z_3 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}\right\} = \quad (\ldots$$

4) Special cases

5) Confluent Hypergeometric function: $zu'' + (c-z)u' - au$

a) $\Phi(a,b;z) = \lim_{b\to\infty} F(a;b;c;z/b)$

b) $z=1 \to z=\infty \quad z=\infty$ not regular

c) Series soln.

d) Functions related to CHGF

$$a = \alpha + \beta + \gamma$$

$$b = \alpha + \beta' + \gamma$$

$$1 - c = \alpha' - \alpha$$

DE for the reduced equation is

$$\overset{-\text{var}}{z(z-1)}\,u'' + [c-(a+b+1)z]\,u' - ab\,u = 0$$

$$z(1-z)$$

$z_2 = \infty$, $z_1 = 0$, $z_3 = 1$ $\quad$ ($\alpha = 0$, $\gamma = 0$, $\alpha' = 1-c$, $\gamma' = c-a-b$
$\beta = a$, $\beta' = b$

$$u'' + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\right)u' + \frac{1}{z(z-1)}\,ab\cdot u$$

$$\cancel{z(z-1)u'' + [\alpha+\alpha'-1 + z(1-\gamma-\gamma'+1-\alpha-\alpha')]\,u + ab\,u = 0}$$

$$u'' + \left(\frac{1-1+c}{z} + \frac{1-c+a+b}{z-1}\right)u' + \frac{ab}{z(z-1)}\,u = 0$$

$$z(z-1)\,u'' + [-c + z(1+a+b)]\,u' + ab\,u = 0$$

$$\Rightarrow \quad \underline{z(1-z)\,u'' + [c - z(1+a+b)]\,u' - ab\,u = 0}$$

Hypergeometric equal

$$\Rightarrow \quad u(z) = F(a,b;c;z)$$

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} F(a;b;c,z')$$

$$\cdot\, F\left(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha',\ \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}\right)$$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1-c & b & c-a-b & \end{Bmatrix}$$

$\Rightarrow$

$$u'' + \left[\frac{1-1+c}{z} + \frac{1-c+a+b}{z-1}\right]u' + ab\,u = 0$$

$$z(z-1)u'' + \left[c(z-1) + z(1-c+a+b)\right]u' + ab\,u = 0$$

$$z(z-1)u'' + \left[-c + z(c+1-c+a+b)\right]u' + ab\,u = 0$$

$$\underline{z(1-z)u'' + \left[c - z(1+a+b)\right]u' - ab\,u = 0}$$

$a = \alpha+\beta+\gamma$

$b = \alpha+\beta'+\gamma$

$1-c = \alpha'-\alpha$

Hypergeometric Function

$$z(1-z)u'' + [c-(a+b+1)z]u' - ab\,u = 0$$

regular singular pts

$z=0$, $r_1=0$, $r_2=1-c$

$z=\infty$ $r_1=a$, $r_2=b$.

$z=1$ $r_1=0$, $r_2=\gamma'-\gamma = c-a-b$.

~~Regular~~ solutions about the regular singular pts.

1) $z=0$, $r_1=0$

$$F(a,b;c;z) = \sum_{n=0}^{\infty} c_n z^n$$

$$c_n = \frac{(a+n-1)(b+n-1)}{n(n+c-1)} c_{n-1} \qquad n=1,2,\dots$$

when c is ~~non-positive~~ not equal to $0,-1,-2,\dots$

we have

$$c_n = \frac{(a+n-1)(b+n-1)}{n(n+c-1)} c_{n-1} = \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\,c(c+1)\cdots(c+n+1)}$$

$$= \frac{\Gamma(a+n)/\Gamma(a)\quad \Gamma(b+n)/\Gamma(b)}{\Gamma(c+n)/\Gamma(c)\quad \Gamma(n+1)} \qquad c_0 = 1.$$

hence

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

This is called the "hypergeometric" series

ex $$F(1,b,b;z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

" Geometric series"

The above series solution is valid up to next singular point $z=1$. Hence

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \qquad |z|<1$$

second soln. $r = r_2 = 1-c$ $\Rightarrow$ 13 den sonraki satırlara bak

If $1-c$ is NOT an integer, a second solution of (1) is of the form $z^{1-c} g_1(z)$ where $g_1(z)$ satisfies

$$z(z-1)g_1'' + [(a+b-2c+3)z + c-2]g_1' + (a-c+1)(b-c+1)g_1 = 0$$

Proof. $u' = (1-c)z^{-c} g_1 + z^{1-c} g_1'$

$u'' = -c(1-c)z^{-c-1} g_1 + 2(1-c)z^{-c} g_1' + z^{1-c} g_1''$

$\cancel{z^{1-c}} g_1'' z(1-z) - c\,z(1-z)\cancel{z^{-c-1}}\, g_1 + 2z(1-z)(1-c)\cancel{z^{-c}} g_1'$

$+ [c-(a+b+1)z](1-c)z^{-c} g_1 + [c-(a+b+1)z]\cancel{z^{1-c}} g_1'$

$- ab\, z^{1-c} g_1 = 0$

$z(1-z)g_1'' + [2(1-z)(1-c) + c - (a+b+1)z]g_1'$

$+ [-\frac{1-z}{z}c(1-c) + \frac{c-(a+b+1)}{z} - ab]g_1 = 0$

$$\cancel{\left[+c(1-c)-ab+\frac{-(1-c)c-(a+b+1)}{z}\right]} g_1$$

$$\cancel{c^2-(a+b+1)}$$

$$\Rightarrow z(1-z)g_1'' - c(1-c)\cancel{\frac{1-z}{z}} g_1 + 2(1-c)(1-z)g_1'$$

$$+(1-c)\left[\cancel{\frac{c-(a+b+1)z}{z}}\right] g_1 + [c-(a+b+1)z]g_1'$$

$$-ab\, g_1 = 0$$

$$z(1-z)g_1'' + [2(1-c)+c-(a+b+1+2-2c)z]g_1'$$

$$+[c(1-c)-(a+b+1)-ab]g_1 = 0$$

$$z(1-z)g_1'' + [2-c-(a+b+3-2c)z]g_1'$$

$$-[a-c+1][b-c+1]\, g_1 = 0$$

$g_1 \sim \cancel{F(a,b,c;z)}$ ✓

$$z(1-z)g_1'' + [2-c-(a-c+1+b-c+1+1)z]g_1'$$

$$-(a-c+1)(b-c+1)\, g_1 = 0$$

$F(a-c+1, b-c+1; 2-c; z)$ ✓

✓ $u(z) = \alpha\, F(a,b;c;z) + \beta\, z^{1-c} F(a-c+1, b-c+1; 2-c; z)$ ✓

Second regular singular point

Similarly

for $z=1$ $\qquad r_1 = 0, \quad r_2 = c-a-b.$

we have if $c-a-b$ = NOT integer

$$u(z) = g_2(1-z) + (1-z)^{c-a-b} g_3(1-z)$$

where g_2 and g_3 power series of $1-z$

for $z=\infty$ $\qquad r_1 = a, \quad r_2 = b$

we have $a-b$ = NOT integer

$$u(z) = \left(\frac{1}{z}\right)^a g_4\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^b g_5\left(\frac{1}{z}\right)$$

where g_4 and g_5 are power series in $(1/z)$.

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,z^n}{\Gamma(c+n)\,n!}$$ ①.

is valid only up to the next singul. point $z=1$ hence the above series expansion is valid for $|z|<1$

ii) $r_2 = 1-c$. we assume that $1-c$ = NOT a non-negative integer.

then $u_2(z) = z^{1-c}\, g_1(z)$

insert it into the HDE

$$z(1-z)u'' + [c-(1+a+b)z]u' - abu = 0$$

we get

$$z(z-1)g_1'' + [c-2+(a+b-2c+3)z]g_1' + (a-c+1)(b-c+1)g_1 = 0$$

or

$$z(1-z)g_1'' + [2-c-(a+b-2c+3)z]g_1' - (a-c+1)(b-c+1)g_1$$

$$\Rightarrow g_1(z) = F(a-c+1,\, b-c+1;\, 2-c;\, z)$$

$1+a+b-2c+2$

$$\Rightarrow u(z) = A\,F(a,b;c;z) + B\,z^{1-c}\,F(a-c+1,\, b-c+1;\, 2-c;\, z)$$

2) about the regular singular pt $z = 1$

$r_1 = 0$, $r_2 = c-a-b$ = NOT an integer.

~~u(t)~~ =

Hypergeometric eqn

$$z(1-z)u'' + [c-(1+a+b)z]u' - ab\,u = 0$$

let $z = 1-y$. $\Rightarrow 1-z = 1-1+y = y$

$u = g_2(y)$

$$(1-y)y\,u_{yy} + [c-(1+a+b)(1-y)]u_y' - ab\,u = 0$$

$$(1-y)y\,u_{yy} + [-c+1+a+b + (1+a+b)y]u_y - ab\,u = 0$$

$$u_1 = g_2(z) = F(a,b;-c+1+a+b;\,1-z)$$

$$u_2 = (1-z)^{c-a-b}\,g_3(1-z)$$

~~y = 1-z~~

$$g_3 = F(c-b,\,c-a;\,1+c-a-b;\,1-z)$$

$$u(z) = A\,F(a,b;-c+1+a+b;\,1-z) + B\,(1-z)^{c-a-b}F(c-b,c-a;1+c-a-b;\,1-z)$$

3) about the regular singular pt $z = \infty$

i) $r_1 = a$ $\quad$ $a - b$ = NOT an integer

$$u_1(z) = z^{-a} g_4(1/z)$$

let $u(z) = z^r g_4(1/z)$ then the HDE becomes

$$y(1-y) g_4'' + [\cancel{1} a - b - 2r - (2 - c - 2r) z] g_4'$$
$$+ [r^2 - ar + rc - \tfrac{1}{z}(r+a)(r+b)] g_4 = 0 \quad)$$

$r = -a$

$$y(1-y) g_4'' + [1+a-b - (2+2a-c) z] g_4'$$
$$+ a(-1-a+c) g_4$$

$$g_4 = F(a, 1+a-c; 1+a-b; \tfrac{1}{z})$$

$$a^2 + a - ac$$
$$-a(-a+1+c$$

Similarly

$$u(z) = A z^{-a} F(a, 1+a-c; 1+a-b; \tfrac{1}{z})$$
$$+ B z^{-b} F(b, 1+b-c; 1+b-a; \tfrac{1}{z})$$

Kummer's solution for F

$$u(z) = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} F\Big(\underbrace{\alpha+\beta+\gamma}_{a},\ \underbrace{\alpha+\gamma+\beta'}_{b};\ \underbrace{1+\alpha-\alpha'}_{c};\ \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}\Big) \quad (15.1)$$

(i) columns of P can be interchanged

6 New solutions

1 2 3	3 1 2
1 3 2	3 2 1
2 1 3	
2 3 1	

6 differ $a \leftrightarrow b$ invariance of F

(ii) Riemann eqn. is invariant for $\alpha \leftrightarrow \alpha'$, $\beta \leftrightarrow \beta'$, $\gamma \leftrightarrow \gamma'$

but (15.1) is invariant here

$\alpha \leftrightarrow \alpha'$, $\gamma \leftrightarrow \gamma'$

gives two transformation for each we have two solns

as a total 4 new solutions.

Hence $4 \times 6 = 24$ new solutions "Kummer's Solutions"

α	γ	α	γ'
α	γ'	α'	γ
α'	γ	α'	γ'
α	γ'	α	γ

4 differ

alternative way to find solution about $z=1$
using the symmetries

Multiply the right handside by $(-1)^{\alpha} z_2^{\alpha+\gamma}$

and interchange the first and last columns.

$z_3 \leftrightarrow z_1$, $\alpha \leftrightarrow \gamma$, $\alpha' \leftrightarrow \gamma'$

$$(-1)^{\alpha} z_2^{\alpha+\gamma} \left(\frac{z-z_3}{z-z_2}\right)^{\gamma} \left(\frac{z-z_1}{z-z_2}\right)^{\alpha} F\left(\alpha+\beta+\gamma,\ \alpha+\gamma+\beta';\ 1+\gamma-\gamma';\ \frac{(z-z_3)(z_1-z_2)}{(z-z_2)(z_1-z_3)}\right)$$

let $z_1=0$, $z_2=\infty$, $z_3=1$. we get

$$z^{\alpha}(-z+1)^{\gamma} F(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\gamma-\gamma';\ 1-z)$$

change $\gamma \to \gamma'$

$$z^{\alpha}(1-z)^{\gamma'} F(\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma';\ 1+\gamma'-\gamma;\ 1-z)$$

$$\cancel{g_2 = z^{\alpha}(1-z)^{\gamma} F(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\gamma-\gamma';\ 1-z)}$$

$\cancel{g_3}$ =

for our case $\alpha=0$, $\alpha'=1-c$, $\beta=a$, $\beta'=b$
$\gamma=0$, $\gamma'=c-a-b$.

about $z=1$

$$u(z) = \alpha \underbrace{F(a, b;\ a+b+1-c;\ 1-z)}_{g_2} + \beta \underbrace{(1-z)^{c-a-b} F(c-b, c-a;\ 1+c-a-b;\ 1-z)}_{g_3}$$

(17)

(19)

$u(t) = z^r w(\frac{1}{z})$

~~$z(1-z)u'' + [c-(1+a+b)z]$~~

$z(1-z)u'' + [c-(1+a+b)z]u' - ab\,u = 0$

$\Rightarrow z(1-z)w'' + [1-a-b-2r-(2-c-2r)z]w'$
$+ [r^2 - r + rc - \frac{1}{z}(r+a)(r+b)]w = 0$

$r = -a, \quad r = -b$

$$u(z) = A\, \underbrace{z^{-a} F(a, 1+a-c; a-b+1; \tfrac{1}{z})}^{g_4} + B\, \underbrace{z^{-b} F(b, 1+b-c; b-a+1; \tfrac{1}{z})}_{g_5}$$

Some special cases

(i) $F(a,b,b;z) = \sum \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} z^n = (1-z)^{-a}$

(ii) $F(\frac{1}{2},\frac{1}{2},\frac{3}{2};z^2) = \frac{1}{z}\sin^{-1}z$ $\quad\left(z\,u(z^2) = \sin^{-1}z \right)$

(iii) $F(1,1;2;-z) = \frac{\ln(1+z)}{z}$ (already)

Jacobi Functions

$(1-z^2)u'' + [b-a-(a+b+2)z]u' + \lambda(\lambda+a+b+1)u = 0$

$P_\lambda^{a,b}$

Convergent

$z = 1-2z'$ $\qquad (1-z)(1+z) = (1-1+2z')(1+1-2z')$

$dz = -2dz'$ $\qquad = 4z'(1-z')$

$u' = u_{z'}(-\frac{1}{2})$

$u'' = \frac{1}{4}u_{z'z'}$

$z'(1-z')u_{z'z'} - \frac{1}{2}[b-a-(a+b+2)(1-2z')]u_{z'} + \lambda(\lambda+a+b+1)u=$

$z'(1-z')u_{z'z'} - [a+1 + (a+b+2)z']u_{z'} + \lambda(\lambda+a+b+1)u = 0$

$F(-\lambda, \lambda+a+b+1, a+1, \frac{1-z}{2})$ Hypergeomet fn

$P_\lambda^{(a,b)} = \frac{\Gamma(\lambda+a+1)}{\Gamma(\lambda+1)\Gamma(a+1)} F(-\lambda, \lambda+a+b+1, a+1, \frac{1-z}{2})$

A special case (Confluent Hypergeometric)

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} \underbrace{P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}}_{y(z)}$$

$$y'' + \left[\frac{2-c}{z-z_1} + \frac{1+a+b}{z-z_2} + \frac{1+c-a-b}{z-z_3}\right]y'$$

$$+\left[\frac{(z_2-z_1)(z_2-z_3)\beta\beta'}{(z-z_2)^2(z-z_4)(z-z_3)}y\right] = 0$$

let $z_1 = 0$, $z_2 = b = 2\alpha$, $z_3 = \alpha$

$$y'' + \left(\frac{2-c}{z} + \frac{1+a+2\alpha}{z-2\alpha} + \frac{1+c-\cancel{a}-2\alpha}{z-\alpha}\right)y'$$

$$+ \frac{2\alpha(\alpha)a2\alpha}{(z-2\alpha)^2 z(z-\alpha)}y = 0$$

$$y'' + \left(\frac{2-c}{z} - 1 + 2\right)y' + \frac{a}{z}y = 0$$

$$zy'' + (z+2-c)y' - ay = 0$$

$$\cancel{F\left(a,b;c;\tfrac{z}{b}\right) = F(a;z\alpha;c}$$

$$u(z) = \Phi(a;c;z) = \lim_{b\to\infty} F(a;b;c;z/b)$$

$z' = zb,\quad z = z'/b$

$$z(1-z)u'' + [c - (1+a+b)z]u' - abu = 0$$

[illegible] $\qquad u' = u_{z'}$

$$\frac{z'}{b}\left(1-\frac{z'}{b}\right)b^{\cancel{2}}\, u_{z'z'} + \cancel{b}\left[c-(1+a+b)\frac{z'}{b}\right]u_{z'} - a\cancel{b}\,u = 0$$

$$z'(1-z'/b)\, u_{z'z'} + [c-(1+a+b)z'/b]u_{z'} - au = 0$$

$$\Rightarrow\quad z' u_{z'z'} + (c - z')u_{z'} - a u = 0 \quad \checkmark$$

Confluent Hypergeometric Function

Consider the case $z_1 = 0$ and $z_2 z_3$

and $\alpha = \gamma = 0$

$$u'' + \left(\frac{1+\alpha+\alpha'}{z} + \frac{1+\beta+\beta'}{z-z_2} + \frac{1+\gamma+\gamma'}{z-z_3}\right)u' \qquad 1-c = \alpha' - \alpha$$

$$+ \beta\beta' \frac{z_2(z_2-z_3)}{(z-z_2)^2 z (z-z_3)} u = 0$$

$$u'' + \left(\frac{\overset{2-c}{c}}{z} + \frac{1-a-b}{z-z_2} + \frac{1-c-a+b}{z-z_3}\right)u' + \frac{ab z_2 (z_2-z_3)}{z(z-z_2)^2(z-z_3)} u = 0$$

let $z_2 = b = 2\alpha\eta$, $z_3 = \alpha\eta$

$$u'' + \left(\frac{\overset{2-c}{c}}{z} + \frac{1-a-2\alpha\eta}{z-2\alpha\eta} + \frac{1-c-a+2\alpha\eta}{z-\alpha\eta}\right)u'$$

$$+ \frac{4\eta^2(\eta)\,a}{z(z-2\eta)^2(z-\underset{\to\eta}{\alpha})} u = 0$$

let $\eta \to \infty$

$$u'' + \left(\frac{c}{z} + 1 - 2\right)u' - \frac{a}{z}u = 0$$

$$zu'' + (\overset{2-c}{c} - z)u' - au = 0$$

$r(r-1)+rc=0$ (21)

$$z u'' + (c-z)u' - au = 0$$

$r_1=0,\ r_2=1-c$

$z=1$ singularity is pushed out to infinity.

$z=0$ and $z=\infty$ (here $z=\infty$ is not anymore a regular singular point it is an irregular singular point).

$$z=\frac{1}{t} \qquad u' = u_t\left(-\frac{1}{t^2}\right),\quad u'' = \frac{2}{t^3}u_t + \frac{1}{t^4}u_{tt}$$

$$\frac{1}{t}t^4 u_{tt} + 2t^3 u_t + \left(c-\frac{1}{t}\right)(-t^2 u_t) - au = 0$$

$$u_{tt} + 2u_t + (-ct+1)\frac{1}{t^2}u_t - \frac{a}{t^3}u = 0$$

$$u_{tt} + \left(2-\frac{c}{t}+\frac{1}{t^2}\right)u_t - \frac{a}{t^3}u = 0$$

$t=0$ is an irregular singular point

Series expansion

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{n=1}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{z^n}{n!}$$

$$= \lim_{b\to\infty} F\left(a,b;c;\frac{z}{b}\right)$$

$$\lim_{b\to\infty}\frac{\Gamma(b+n)}{b^n\,\Gamma(b)} = 1 \qquad \text{prove by induction}$$

Confluent Hypergeometric function for $r = 1-c \neq$ Not an integer

$$z u'' + (c-z) u' - a u = 0$$

$$u = z^{1-c} g$$

$$u' = (1-c) z^{-c} g + z^{1-c} g'$$

$$u'' = -c(1-c) z^{-c-1} g + 2(1-c) z^{-c} g' + z^{1-c} g''$$

$$-c(1-c) z^{-c} g + 2(1-c) z^{\cancel{1-c}} g' + z^{2-c} g''$$
$$+ (c-z)(1-c) \underline{z^{-c}} g + (c-z) \cancel{z}^{1-c} g' - a \underline{z^{1-c} g} = 0$$

$$z^2 g'' + [2(1-c) z + (c-z) z] g'$$
$$+ [\cancel{-c(1-c)} + \cancel{c(1-c)} - z(1-c) - a z] g = 0$$

$$z^{\cancel{2}} g'' + \cancel{z}(2 - 2c + c - z) g' - \cancel{z}(1-c+a) g = 0$$

$$z g'' + (2-c-z) g' - (1-c+a) g = 0$$

$$u = z^{1-c} \Phi(1-c+a, 2-c, z)$$

Gen. Soln.

$$u(z) = \alpha \Phi(a, c; z) + \beta z^{1-c} \Phi(1-c+a, 2-c, z)$$

$1-c$ Not an integer or 0

This is the solution analytic about $z=0$

$r_1 = 0$, "confluent hypergeometric function"

for $r_2 = 1-c$ can be found by symmetry

since the next singular point is at $z=\infty$
then the "confluent hypergeometric function"
is an "entire function"

let $u = z^{1-c} v$

$u' = (1-c) z^{1-c-1} v + z^{1-c} v'$

$u' = (1-c) z^{-c} v + z^{1-c} v'$

$u'' = -c(1-c) z^{-c-1} v + 2(1-c) z^{-c} v' + z^{1-c} v''$

$z^{2-c} v'' + 2(1-c) \cancel{z^{1-c}} v' - c(1-c) z^{-c} v$

$+ (c+z)(1-c) z^{-c} v + (c-z) \cancel{z^{1-z}} v'$

$- a \cancel{z^{1-c}} v = 0$

$z v'' + [2(1-c) z + \cancel{c(1-c)} \; c - z] v'$

$+ [-a - c(1-c)\cancel{\tfrac{1}{z}} + c(1-c)\cancel{\tfrac{1}{z}} - (1-c)] v = 0$

$z v'' + (\cancel{c - (1-z+c)} z] v' + [-a-1+c] v = 0$

$(2 - 2c + c - z)$

$2 - c - z$

$$z v'' + \{ c - (-1+2c) z \} v' + [c-a-1] v = 0$$

$$\cancel{z = \alpha z'} \qquad z' = \alpha z$$

$$\cancel{\alpha z} \quad \frac{1}{\alpha} z' \, v_{z'z'} \alpha^2 + \left[c - (-1+2c) \frac{z'}{\alpha} \right] \alpha v_{z'} + [c-a-1] v$$

$$z' v_{z'z'} + \left[\frac{c}{\cancel{\alpha}} - (-1+2c) \frac{z'}{\alpha} \right] v_{z'} + \frac{c-a-1}{\alpha} v = 0$$

$$\alpha = \frac{1}{2c-1}$$

$$v = \Phi\left(\frac{1-a-c}{2c-1}, c; \frac{z}{2c-1} \right)$$

$$u = z^{1-c} \Phi\left(\frac{1-a-c}{2c-1}, c; \frac{z}{2c-1} \right).$$

$$u = A\, \Phi(a, b; z) + B z^{1-c} \Phi\left(\frac{1-a-c}{2c-1}, c; \frac{z}{2c-1} \right)$$

$1-c$ NOT integer.

Functions related to the confluent hypergeometric Funct

Hermite and Laguerre functions

. Hermite funct

$$u'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)u = 0$$

let $u(z) = e^{-\frac{1}{4}z^2} v(z)$

$$u' = -z\tfrac{1}{2}e^{-\frac{1}{2}z^2}v + e^{-\frac{1}{2}z^2}v'$$

$$u'' = -\tfrac{1}{2}e^{-\frac{1}{2}z^2}v + \tfrac{z^2}{4}e^{-\frac{1}{2}z^2}v - z e^{-\frac{1}{2}z^2}v' + e^{-\frac{1}{2}z^2}v''$$

$$v'' - zv' + \cancel{\tfrac{z^2}{4}v} - \tfrac{1}{2}v + (\nu + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{4}z^2})v = 0$$

$\cancel{v'' + (-z \cdot (+\nu - \tfrac{1}{4})z}$ $\quad v'' - zv' + (\nu - \tfrac{1}{4})v = 0$

$v'' - zv' + \nu v = 0$

$$v = \Phi(\tfrac{1}{4} - a, 0; z)$$

$(z\rho')'$

$z\rho'' = (z\rho')' - \rho'$

$$\rho''' - z\rho'' + (\nu - \tfrac{1}{4})\rho' = 0$$

$$\rho'' - z\rho' - \rho$$

$$u(z) = e^{-\frac{1}{4}z^2} v(z^2/2) \qquad \xi = z^2/2$$

$$u_z = -\frac{1}{2} z e^{-\frac{1}{4}z^2} v + z e^{-\frac{1}{4}z^2} v_\xi$$

$$u_{zz} = -\frac{1}{2} e^{-\frac{1}{4}z^2} v + \frac{1}{4} z^2 e^{-\frac{1}{4}z^2} v + e^{-\frac{1}{4}z^2} v_\xi - \frac{z^2}{2} e\, v_\xi$$
$$+ z^2 e^{-\frac{1}{4}z^2} v_{\xi\xi}$$

$$2\xi v_{\xi\xi} + (-2\xi + 1) v_\xi + (\nu v) = 0$$

$$\boxed{\xi v_{\xi\xi} + \left(\frac{1}{2} - \xi\right) v_\xi + \frac{\nu}{2} v = 0}$$

$$v = \Phi\left(-\frac{\nu}{2}, \frac{1}{2}, z^2/2\right)$$

$$u(z) = e^{-\frac{1}{4}z^2} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}, z^2/2\right) \checkmark$$

- with [illegible] normalised parameter

$$\phi = \mathrm{Erf}(z) = \int^{z} e^{-t^2} dt = z\,\Phi(\tfrac{1}{2}, \tfrac{3}{2}; -z^2) \Rightarrow \text{arkada}$$

Prove this $\quad \cancel{\phi' = e^{-z^2}}, \quad \cancel{\phi'' = -2ze^{-z^2}}$

$$\phi = \frac{1}{\sqrt{z}}\,\mathrm{erf}(\sqrt{z})$$

$$\phi_z = e^{-z^2}$$

$$\phi_{zz} = -2z\,\phi_z$$

Bessel Function

$$u'' + \frac{1}{z}u' + \left(1 - \frac{\nu^2}{z^2}\right)u = 0$$

letting

$$u(z) = z^{\nu} e^{-iz}\, v(2iz), \quad \xi = 2iz$$

$$v(\xi) = \Phi(\nu + \tfrac{1}{2}, 2\nu + 1, \xi)$$

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu} e^{-iz}\,\Phi(\nu + \tfrac{1}{2}, 2\nu+1; \xi)$$

$$\phi = z\psi$$

$$\phi' = \psi + z\psi'$$

$$\phi'' = 2\psi' + z\psi'' = -2z(\psi + z\psi')$$

$$z(1 + 2z)\psi'' \cancel{+ 2(1+z)\psi}$$

SET 7

MATH 543: FUCHSIAN DIFFERENTIAL EQUATIONS HYPERGEOMETRIC FUNCTION

References: DK and Sadri Hassan.

Historical Notes: Please read the book *Linear Differential Equations and the Group Theory* by *Jeremy J. Gray , Birkhouser, 2000* for the contributions of **Euler, Pfaff , Gauss, Riemann, Kummer, Jacobi** and others on the P symboll and on the Hypergeometric equation and hyoergeometric function.

Definition 1. Linear ordinary differential equations having only regular singular points are called *Fuchsian Differential Equations* (FDE).

FDE with two regular singular points

Definition 2. (Regular singular point at ∞). If the transformed DE by $z = \frac{1}{t}$ has regular singular point at $t = 0$ then the original DE has a regular singular point (in the extended complex plane) at $z = \infty$.

Proposition 1. *Second order FDE having only two singular points is equivalent to a DE with constant coefficients, hence solvable in terms of the elementary functions* $\sin z$ *,*$\cos z$ *and polynomial in* z*.*

FDE with three regular singular points: The Riemann equation and Riemann P Symbol.

Riemann has put a second order linear FDE with three regular singular points $z = z_1$, $z = z_2$ and $z = z_3$ into the form

$$u'' + \left(\frac{1+\alpha+\alpha'}{z-z_1}\frac{1+\beta+\beta'}{z-z_2} + \frac{1+\gamma+\gamma'}{z-z_3}\right) + \left(\frac{(z_1-z_2)(z_1-z_3)\alpha\,\alpha'}{z-z_1} + \frac{(z_2-z_1)(z_2-z_3)\beta\beta'}{z-z_2} + \frac{(z_3-z_1)(z_3-z_1)\gamma\,\gamma'}{z-z_3}\right)\frac{u}{(z-z_1)(z-z_2)(z-z_3)} = 0, \quad (1)$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are constant satisfying the constraint

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

The solution of (1) is denoted as the Riemann P-symbol

$$u(z) = P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} \tag{2}$$

The columns in the P symbol indicate the location of the regular singular points and the corresponding roots of the indical equation, i.e.,

1. $z = z_1$, $r_1 = \alpha$ and $r_2 = \alpha'$.
2. $z = z_2$, $r_1 = \beta$ and $r_2 = \beta'$.
3. $z = z_3$, $r_1 = \gamma$ and $r_2 = \gamma'$.

One can reduce the 9 number of parameters in the Riemann equation (or in the Riemann P symbol) into three parameters by using the following type of transformation

(i) $u(z) = (z - z_1)^r (z - z_2)^s (z - z_3)^t v(z), \quad r + s + t = 0$
(ii) The Mobius transformation: $z' = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$

we obtain first (from (i) above)

$$P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z - z_1}{z - z_2}\right)^{\alpha} \left(\frac{z - z_3}{z - z_2}\right)^{\gamma} P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ 0 & a & 0 & z \\ 1 - c & b & c - a - b & \end{array} \right\} \tag{3}$$

and the using (ii) we obtain

$$P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z - z_1}{z - z_2}\right)^{\alpha} \left(\frac{z - z_3}{z - z_2}\right)^{\gamma} P \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1 - c & b & c - a - b & \end{array} \right\}, \tag{4}$$

where

$$
\begin{aligned}
z' &= \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}, \quad a = \alpha+\beta+\gamma, & (5)\\
b &= \alpha+\beta'+\gamma, \quad c = 1+\alpha-\alpha' & (6)
\end{aligned}
$$

The P symboll in the right hand side of (4) is the hypergeomtric function, $F(a,b;c;z')$

$$
F(a,b;c;z) = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\} \tag{7}
$$

Problems

1. Prove that the hypergeometric function defined above (7) satisfies the differential equation (hypergeometric DE)

$$
z(1-z)\,u'' + [c-(1+a+b)\,z)\,u' - ab\,u = 0 \tag{8}
$$

2. Find (Kummer) transformations leaving the Reiemann equation form invariant.

3. Prove that

$$
F = \sum_{n=0}^{\infty} \frac{a_n\, b_n}{c_n}\, \frac{z^n}{n!}
$$

where

$$
a_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad b_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \quad c_n = \frac{\Gamma(c+n)}{\Gamma(c)} \quad |z| \leq 1
$$

4. Prove the following Proposition.

Proposition 2. *Solutions of the hypergeometric function bout its regular singular points $z=0$, $z=\infty$ and $z=1$, provided $1-c$, $b-a$, and $c-a-b$ are not integers, are respectively given by*

$$
\begin{aligned}
u(z) &= A_1 F(a,b;c;z) + B_1 z^{1-c} F(b-c+1, a-c+1; 2-c; z), & (9)\\
u(z) &= A_2 z^{-a} F(a, a-c+1; a-b+1; \frac{1}{z}) + B_2 z^{-b} \\
&\quad F(b, b-c+1; b-a+1; \frac{1}{z}), & (10)\\
u(z) &= A_3 F(a,b;a+b+1-c;1-z) + B_3 z^{c-a-b} F(c-b, c-a; \\
&\quad 1+c-a-b; 1-z), & (11)
\end{aligned}
$$

where $A_i, B_i, (i = 1,2,3)$ *are arbitrary constants.*

5. Find the solutions (10) and (11) about the regular singular points $z = \infty$ and $z = 1$ by the use of *Kummer's* transformations mentioned in the Pr.2.

6. Prove the following: The Jacobi function $P_\lambda^{(\alpha,\beta)}$ satisfying the Jacobi equation equation

$$(1-z^2)u'' + [\beta - \alpha - (\alpha+\beta+2)z]u' + \lambda(\lambda+\alpha+\beta+1)u = 0$$

can be written in terms of the hypergeometric function

$$P_\lambda^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} F(-\lambda, \lambda+\alpha+\beta; \alpha+1; \frac{1-z}{2})$$

When $\lambda = n$ a non-negative integer the the solution becomes Jacobi polynomials containing the Legendre, and Tchbechev polynomials.

7. Prove that when the regular singular points $z = z_2$ and z_3 of the Riemann equation are pushed out ∞ then the resulting function function is the confluent hypergeometric function $\Phi(a,c;z)$ in

8. Prove that $\Phi(a,c,z) = \lim_{b\to\infty} F(a,b;c;\frac{z}{b})$

9. Prove that the confluent hypergeometric function satisfies the DE

$$z\,u'' + (c-z)\,u' - a\,u = 0 \qquad (12)$$

10. Prove that the point $z = \infty$ of the confluent hypergeometric equation (12) is an irregular singular point.

11. Find solutions of the confluent hypergeometric equation about all it regular singular points. The form of the equation is of the form

$$u(z) = A\,\Phi(a, c; z) + B\,z^{1-c}\,\Phi(a', c', b'\,z)$$

where A and B are arbitrary constants. Find the constants a', c' and b'

12. Prove that Bessel's function $J_v(z)$ satisfying the Bessel equation $u'' + \frac{1}{z}u' + (1 - \frac{\nu^2}{z^2})u = 0$ is given by

$$J_v(z) = \frac{1}{\Gamma(\nu+1)}(z/2)^{\nu}\,\Phi(\nu + 1/2, 2\nu + 1; 2iz)$$

13. Prove that

$$\frac{d^n}{dz^n}F(a, b; , c; z) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)}\,F(a+n, b+n; c+n; z)$$

Use induction.

14. Prove the following:

(i) $F(-a, b; b; -z) = (1+z)^a$,

(ii) $F(1, 1; 2; -z) = \frac{1}{z}\ln(1+z)$,

(iii) $F(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2) = \frac{1}{z}\sin^{-1} z$,

(iv)

$$F(\frac{1}{2}, \frac{1}{2}; 1; z^2) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - z^2\sin^2\theta}},$$

(v) $e^z = \lim_{b\to\infty} F((1, b, 1, \frac{z}{b})$

(vi) The error function

$$Erf(z) = \int_0^{\infty} e^{-t^2}\,dt$$

15. Prove that the Hermite-Weber differential equation

$$u'' + (\nu + \frac{1}{2} - \frac{1}{4}z^2)u = 0$$

can be converted to the confluent hypergeometric equation by

$$u(z) = e^{\frac{-z^2}{4}}\,v(\xi), \quad \xi = \frac{z^2}{2}$$

with

$$v(\xi) = \Phi(-\nu/2, 1/2; \xi)$$